ON PURELY LOXODROMIC ACTIONS

ILYA KAPOVICH

Abstract. We construct an example of an isometric action of $F(a,b)$ on a δ-hyperbolic graph Y, such that this action is acylindrical, purely loxodromic, has asymptotic translation lengths of nontrivial elements of $F(a,b)$ separated away from 0, has quasiconvex orbits in Y, but such that the orbit map $F(a,b) \to Y$ is not a quasi-isometric embedding.

1. Introduction

There are many natural situations in geometric topology and geometric group theory when one wants to understand, given a group G acting on some Gromov-hyperbolic space X, and a finitely generated "purely loxodromic" subgroup $H \le G$, whether the orbit map $H \to X$ is a quasi-isometric embedding. Here "purely loxodromic" means that every element $h \in H$ of infinite order acts loxodromically on X. The model example of this problem comes from studying subgroups of mapping class groups. Let S be a closed oriented hyperbolic surface and let $\mathcal{C}(S)$ be the curve complex of S (known to be Gromov-hyperbolic by a result of Masur and Minsky [29]). It is known that an element g of the mapping class group $Mod(S)$ acts loxodromically on $\mathcal{C}(S)$ if and only if g is pseudo-Anosov. A finitely generated subgroup $H \le Mod(S)$ is called *convex cocompact* (see [13, 16, 22, 23]) if the orbit map $H \to \mathcal{C}(S)$ is a quasi-isometric embedding. An important open problem in the study of mapping class groups asks whether every "purely pseudo-Anosov" (that is purely loxodromic for the action on $\mathcal{C}(S)$) finitely generated subgroup of $Mod(S)$ is convex cocompact.

Note that if G is a word-hyperbolic group acting by translations on its Cayley graph X, then $g \in G$ is loxodromic if and only if g has infinite order. In this case whenever $H \le G$ is a finitely generated subgroup which is not quasiconvex in G, then H is purely loxodromic but the orbit map $H \to X$ is not a quasi-isometric embedding. However, in this case the orbit of H in X is not a quasiconvex subset of X. Moreover, for a finitely generated subgroup $H \le G$ the orbit map $H \to X$ is a quasi-isometric embedding if and only if every (equivalenty, some) orbit of H in X is quasiconvex. There are many examples of finitely generated (even word-hyperbolic) subgroups of word-hyperbolic groups that are not quasiconvex. For instance, if G is the fundamental group of a closed hyperbolic 3-manifold M fibering over the circle with fiber S, then $G = \pi_1(M)$ is word-hyperbolic and $\pi_1(S) \le G$ is not quasiconvex.

There are some situations where purely loxodromic subgroups do have quasi-isometric embedding orbit maps. Thus a recent paper [25] of Koberda, Mangahas, and Taylor provides a result of this kind. Given a right-angled Artin group $G = A(\Gamma)$ defined by a finite graph Γ, there is an associated Gromov-hyperbolic graph Γ^e (see [24]), called the "extension graph", which comes equipped with a natural isometric action of G. They prove in [25] that for a finitely generated subgroup $H \le G$ the orbit map $H \to \Gamma^e$ is a quasi-isometric embedding if and only if the action of H on Γ^e is purely loxodromic. This result is proved in [25] in the context of exploring a strong form of quasiconvexity for finitely generated subgroups of finitely generated groups called "stability".

The group $Out(F_N)$ (where F_N is a free group of finite rank $N \ge 3$) has a natural isometric action on the "free factor graph" $\mathcal{F}_N$, which is known to be Gromov-hyperbolic [2, 20, 18] and provides one of several $Out(F_N)$ analogs of the curve complex. It is known [2] that $\varphi \in Out(F_N)$ acts on $\mathcal{F}_N$ loxodromically if and only if φ is fully irreducible. There are two types of fully irreducible elements of $Out(F_N)$: atoroidal ones (which have no nontrivial periodic conjugacy classes in F_N and have word-hyperbolic mapping torus groups)

2010 *Mathematics Subject Classification.* Primary 20F65, Secondary 57M.

The author was partially supported by the NSF grant DMS-1405146.

and non-atoroidal ones. It is known [3] that a non-atoroidal $\varphi \in Out(F_N)$ is fully irreducible if and only if φ is induced by a pseudo-Anosov homeomorphism of a compact surface with one boundary component. In [11] Dowdall and Taylor proved that if a finitely generated $H \le Out(F_N)$ is "purely atoroidal" and has the orbit map $H \to \mathcal{F}_N$ being quasi-isometric embedding (which implies that H is also purely loxodromic for the action on $\mathcal{F}_N$) then the natural extension G_H of F_N by H is word-hyperbolic. Hamenstadt and Hensel [17] suggested to call a finitely generated subgroup $H \le Out(F_N)$ "convex cocompact" if the orbit map $H \to \mathcal{F}_N$ is a quasi-isometric embedding. However, with this definition, an infinite cyclic $H = \langle \varphi \rangle \le Out(F_N)$, generated by a non-atoroidal fully irreducible φ, is considered convex cocompact, although the group G_H is not word-hyperbolic in this case. Mann and Reynolds [28] defined a further coarsely Lipschitz coarsely equivariant quotient $\mathcal{P}_N$ of $\mathcal{F}_N$ such that $\mathcal{P}_N$ is Gromov-hyperbolic and such that $\varphi \in Out(F_N)$ acts loxodromically on $\mathcal{P}_N$ if and only if φ is an atoroidal fully irreducible. In a new paper [12] Dowdall and Taylor show that if $H \le Out(F_N)$ is a finitely generated purely atoroidal subgroup such that the orbit map $H \to \mathcal{F}_N$ is a quasi-isometric embedding (so that H is purely loxodromic for the action on $\mathcal{P}_N$) then the orbit map $H \to \mathcal{P}_N$ is also a quasi-isometric embedding. This result provides another interesting example where a purely loxodromic action can be shown to have the orbit map being a quasi-isometric embedding (under the initial assumption that the orbit map $H \to \mathcal{F}_N$ is a quasi-isometric embedding.)

The goal of this note is to show that even if we make rather strong additional geometric assumptions about a purely loxodromic isometric action of a word-hyperbolic group H on a Gromov-hyperbolic space X (including discreteness and quasi convexity of H-orbits), that is not enough to ensure that the orbit map $H \to X$ is a quasi-isometric embedding.

Before stating the main result, we recall several definitions.

Definition 1.1 (Asymptotic translation length)**.** Let G be a group acting isometrically on a metric space X. For an element $g \in G$ the *asymptotic translation length* $||g||_X$ of g on X is

$$||g||_X := \lim_{n\to\infty} \frac{d_X(x, g^n x)}{n},$$

where $x \in X$ is a basepoint.

It is well-known that the above limit always exists and does not depend on the choice of $x \in X$. Moreover, for an element $g \in G$, the map $\mathbb{Z} \to X$, $n \mapsto g^n x$, is a quasi-isometric embedding if and only if $||g||_X > 0$. In particular, if X is Gromov-hyperbolic, then $g \in G$ acts logodromically on X if and only if $||g||_X > 0$.

Definition 1.2 (Acylidrical actions)**.** An isometric action of a group G on a Gromov-hyperbolic space X is said to be *acylindrical* if for every $R \ge 0$ there exist $L \ge 1$ and $M \ge 1$ such that whenever $x, y \in X$ are such that $d_X(x, y) \ge L$ then

$$\#\left(\{g \in G | d_X(x, gx) \le R, d_X(y, gy) \le R\}\right) \le M$$

Acylidrical actions on hyperbolic spaces play a crucial role in studying various generalizations of relatively hyperbolic groups, particularly the so-called acylindrically hyperbolic groups (see, for example [9, 32, 19, 15, 33]), and in the study of group actions on $\mathbb{R}$-trees (see, for example, [10, 21, 34, 1]). The action of $Mod(S)$ on the curve complex $\mathcal{C}(S)$ is also known to be acylindrical, see [6] and this fact has many useful consequences in the study of mapping class groups. Acylindricity is a rather strong assumption, which brings some degree of finiteness to non-proper actions and also imposes substantial algebraic restrictions on the situation.

Out main result is (c.f. Theorem 4.5 below):

Theorem A. There exists a Gromov-hyperbolic graph Y with a simplicial isometric action of $F(a, b)$ on Y such that the following hold:

1. The action of $F(a, b)$ on Y is acylindrical.
2. The action of $F(a, b)$ on Y is purely loxodromic, that is, every $1 \ne g \in F(a, b)$ acts on Y as a loxodromic isometry.
3. For every $1 \ne g \in F(a, b)$ we have $||g||_Y \ge 1/7$.
4. For any $p \in Y$ the orbit $F(a, b)p \subseteq Y$ is a quasiconvex subset of Y.

(5) There exists $C \geq 1$ such that for any $x, y \in F(a,b)$ if $\alpha_{x,y}$ is a geodesic from x to y in the Cayley graph of $F(a,b)$ with respect to the basis $\{a,b\}$, and if $\beta = [x,y]_Y$ is a geodesic from x to y in Y, then α and β are C-Hausdorff close in Y.

(6) For any $p \in Y$, the orbit map $F(a,b) \to Y$, $g \mapsto gp$, is not a quasi-isometric embedding, and, moreover, the action of $F(a,b)$ on Y is not metrically proper.

Note that, by the standard Milnor-Svarc argument (c.f. [8, Proposition 8.19]), if G is a group acting by isometries on a Gromov-hyperbolic metric space X with quasiconvex orbits and if the action is metrically proper (that is, if for every metric ball B the set $\{g \in G | B \cap gB \neq \varnothing\}$ is finite), then G is finitely generated and the orbit map $G \to X$ is a quasi-isometric embedding.

An instructive example for comparison with Theorem A comes from group actions on $\mathbb{R}$-trees that live in the boundary of the Culler-Vogtmann Outer space. Let $\varphi \in \mathrm{Out}(F(a,b,c))$ be an atoroidal fully irreducible automorphism and let $T = T_\varphi$ be the "stable" $\mathbb{R}$-tree for φ, constructed from a train-track representative of φ (see [4, 5] for the construction of T_φ). Then $F_3 = F(a,b,c)$ acts on T freely, isometrically and with dense orbits in T (see, for example, [14, 26]), so that this action is purely loxodromic and all F_3-orbits are quasiconvex in T. Condition (5) of Theorem A also holds in this case because of the so-called "bounded back-tracking property" for "very small" actions of free groups on $\mathbb{R}$-trees [14]. Since the action on T has dense orbits, the set of asymptotic translation lengths of nontrivial elements of F_3 is not separated away from 0. The action is also not acylindrical. Indeed, take $R = 1$. Then for any $M \geq 1$ there exists an element $g \in F_3$ with $0 < ||g|| < 1/M$. Consider the axis $L(g) \subseteq T$, so that g acts on $L(g)$ by a translation of magnitude $||g||_T$. For any $L \geq 1$ take points $x, y \in L(g)$ with $d_T(x,y) \geq L$. Then for $k = 0, 1, 2 \dots, M$ the element g^k translates each of x, y by $k||g||_T \leq 1$ so that we have $\geq M+1$ distinct elements displacing each of x, y by ≤ 1. Thus the action of F_3 on T is indeed not acylindrical. Finally, the orbit map $F_3 \to T$ is not a quasi-isometric embedding. Thus this example satisfies properties (2), (4), (5) and (6) from Theorem A but does not satisfy properties (1) and (3).

Theorem A shows that even very strong additional assumptions on a purely loxodromic action (including discreteness, acylindricity, having quasiconvex orbits and having asymptotic translation lengths of loxodromic elements being separated away from 0) are, in general, not sufficient to imply that the orbit map is a quasi-isometric embedding.

We briefly describe the construction of Y in Theorem A here. We start with an infinite sequence $v_n(a,b) \in F(a,b)$ (where $n = 1, 2, \dots$) of distinct positive 7-aperiodic words, that is such that no v_n contains a subword of the form u^7 for any nontrivial u. We put $w_n = v_n(a,b)c \in F(a,b,c)$. Let K be the set of all positive words $z \in F(a,b,c)$ such that z is a subword of w_n^m for some $m, n \geq 1$. Note that $\{a,b,c\} \subseteq K$. Then Y is the Cayley graph of $F(a,b,c)$ with respect to the generating set K. One can also view Y as a "coned-off" version of the Cayley graph X of $F(a,b,c)$ with respect to $\{a,b,c\}$ where for every $n \geq 1$ and for every conjugate w_n' of w_n in $F(a,b,c)$ we "cone-off" the axis $L(w_n') \subseteq X$ of w_n' in X. See Definition 2.3 below for details. The fact that we are coning off a collection of uniformly quasiconvex subsets of a hyperbolic graph X implies (by [20, Proposition 2.6]) that Y is Gromov-hyperbolic and that part (4) of Theorem A holds. Part (4) in turn easily implies part (3) since $F(a,b) \leq F(a,b,c)$ is a quasiconvex (even convex for X) subgroup. It is also clear from the construction that the orbit map $F(a,b) \to Y$, $g \mapsto gp$, is not a quasi-isometric embedding, and that in fact the action of $F(a,b)$ on Y is not proper.

To see that the action of $F(a,b)$ on Y is purely loxodromic and that has the asymptotic translation length of nontrivial elements of $F(a,b)$ bounded below by $1/7$, we develop a precise formula for computing distances in Y and exploit the 7-aperiodicity property of the words $v_n(a,b)$. Note that the action of $F(a,b,c)$ on X is acylindrical, but we are coning off a collection of subsets of X that are uniformly quasiconvex but are not "geometrically separated" in the sense of [9]. The reason is that the axes of conjugates of distinct w_n and w_m in X can have arbitrarily long overlaps as $n, m \to \infty$. Thus we cannot use the general result, given by Proposition 5.40 of [9], to conclude that the action of $F(a,b,c)$ on Y is acylindrical (which may still be true). Instead we give a direct argument, again exploiting the properties of periodic and aperiodic words in free groups, that the action of $F(a,b)$ on Y is acylindrical. It would be interesting to understand whether, when starting with an acylindrical G-action on a Gromov-hyperbolic space, coning-off a G-equivariant collection

of uniformly quasiconvex subsets (perhaps with appropriate extra assumptions on various constants) always produces an acylindrical action of G on the coned-off space.

I am grateful to Chris Leininger, Paul Schupp, Denis Osin, Michael Hull, Funda Gultepe, Kasra Rafi, Samuel Taylor and Spencer Dowdall for useful discussions.

2. Construction and basic properties of the graph Y

Let $F_3 = F(a,b,c)$ and let X be the Cayley graph of F_3 with respect to the free basis $A = \{a,b,c\}$.

For a word v in some alphabet, we denote by $|v|$ the length of v. For an element $g \in F(a,b,c)$ we denote by $|g|_A$ the freely reduced length of g with respect to A and denote by $||g||_A$ the cyclically reduced length of g with respect to A. Note that $||g||_A = ||g||_X$, the asymptotic translation length for the action of g on X.

When dealing with words over the alphabet $A^{\pm 1}$, we will use $\equiv$ to indicate graphical equality of such words and we will use $=$ to indicate that the words represent the same element of $F(a,b,c)$.

We say that a freely reduced word $v \in F(a,b,c)$ is 7-*aperiodic* if there does not exist a nontrivial cyclically reduced word $u \in F(a,b,c)$ such that u^7 is a subword of v. It is well-known that there exist infinite 7-aperiodic subsets of $F(a,b)$. For a sample reference we can use a result of Ol'shanskii, Lemma 1.2 in [31], where an infinite 7-aperiodic set with additional small cancellation properties is constructed:

Proposition 2.1. [31, Lemma 1.2] *There exists a sequence $v_n(a,b) \in F(a,b)$, where $n = 1,2,3,\dots$ of positive words v_n in $F(a,b)$ with the following properties:*

(1) We have $|v_n| \to \infty$ as $n \to \infty$ and $|v_n| \neq |v_m|$ whenever $m \neq n$.
(2) Each v_n is 7-aperiodic.
(3) If u is a subword of some v_n with $|u| \geq |v_n|/1000$ then u occurs as a subword in v_n exactly once, and u does not occur as a subword of any v_m with $m \neq n$.

Although we don't actually use part (3) of the above proposition in this paper, we record part (3) since it may be useful for further sharpening of the results obtained here.

Convention 2.2. From now and for the remainder of the paper, we fix a sequence of positive words $v_n \in F(a,b)$ satisfying the conclusions of Proposition 2.1.

For $n = 1,2,3,\dots$ put $w_n := v_n c \in F(a,b,c)$.

Note that the words v_n, w_n are positive and thus are freely and cyclically reduced.

Definition 2.3 (The graph Y)**.** Let $v_n \in F(a,b), w_n \in F(a,b,c)$, where $n = 1,2,3\dots$ be as in Convention 2.2. We define a graph Y as follows.

The graph X is a subgraph of Y and $VY = VX$. The extra edges added to X to obtain Y are defined as follows:

For every $n \geq 1$ and every conjugate w_n' of w_n in $F(a,b,c)$ we take the line $L(w_n') \subseteq X$ to be the axis of w_n' when acting on X; for every pair of vertices $x, y \in L(w_n')$ such that $d_X(x,y) \geq 2$ we add an edge joining x and y. We call edges of $Y - X$ *special edges*.

Since X is the Cayley graph of $F(a,b,c)$, every oriented edge e of X already has a label $\mu(e) \in A^{\pm 1}$. If e is an oriented edge of $Y - X$ from a vertex $x \in VX$ to a vertex $y \in VX$, then $x, y \in F(a,b,c)$ and the geodesic segment $[x,y]_X$ is labelled by the freely reduced form z of the element $x^{-1}y \in F(a,b,c)$. We then put $\mu(e) := z$ and $\mu(e^{-1}) = z^{-1}$.

Thus Y is a labelled graph where every oriented edge e of Y has a label $\mu(e)$ which is a nontrivial freely reduced word in $F(a,b,c)$. This assignment satisfies $\mu(e^{-1}) = \mu(e)^{-1}$. Moreover, every special oriented edge e of Y is labelled by some nontrivial subword of some w_n^m.

We equip Y with the simplicial metric d_Y. Note that the set of lines $L(w_n')$, as $n = 1,2,3,\dots$ and w_n' varies over all conjugates of w_n in $F(a,b,c)$, is $F(a,b,c)$-invariant. Hence the translation action of $F(a,b,c)$ on X naturally extends to an action of $F(a,b,c)$ on Y by graph automorphisms, and thus by d_Y-isometries.

If $\gamma = e_1, e_2, \dots, e_k$ is an edge-path in Y, we put $\mu(\gamma) \equiv \mu(e_1)\dots\mu(e_k) \in (A^{\pm 1})^*$. Note that the label $\mu(\gamma)$ need not be a freely reduced word even if the path γ is a geodesic in Y.

Note that the space X is Gromov-hyperbolic, and line each $L(w'_n) \subseteq X$ is a 0-quasiconvex subset of X. Therefore the following statement is a direct corollary of Proposition 2.6 of [20] (see also Proposition 7.12 in [7] for a related statement):

Proposition 2.4. *There exist integer constants $\delta \geq 1$ and $C \geq 1$ such that:*

(1) The space (Y, d_Y) is δ-hyperbolic.
(2) For any $x, y \in X$, if $\alpha = [x,y]_X$ is a d_X-geodesic from x to y in X and $\beta = [x,y]_Y$ is a d_Y-geodesic from x to y in Y then α and β are C-Hausdorff close with respect to d_Y.

Convention 2.5. For the remainder of the paper, we fix a number $C \geq 1$ satisfying the conclusion of Proposition 2.4.

We record the following useful immediate corollary of part (2) of Proposition 2.4:

Corollary 2.6. *Let $x, y \in VX$ and let x' be a vertex of X such that $x' \in [x,y]_X$. Then*

$$|d_Y(x,x') + d_Y(x',y) - d_Y(x,y)| \leq 2C.$$

Proposition 2.7. *For any point $x \in Y$, the orbit $F(a,b)x \subseteq Y$ is a quasiconvex subset of Y*

Proof. We may assume that $x = 1 \in F(a,b)$, so that $F(a,b)x = F(a,b) \subseteq VY$.

Let $g \in F(a,b)$ be arbitrary and let $\alpha = [1,g]_X$ be the (unique) d_X-geodesic path from 1 to g in X. Thus γ is labbeled by the freely reduced $v(a,b)$ form of g. Let $\beta = [1,g]_Y$ be a d_Y-geodesic from 1 to g in Y.

By Proposition 2.4, for every point $p \in \beta$ there exists a vertex q on α such that $d_Y(p,q) \leq C+1$. Thus q represents an elememnt of $f(a,b,c)$ given by some initial segment of the word $v(a,b)$ and hence $q \in F(a,b)$. This shows that $F(a,b)$ is a $(C+1)$-quasiconvex subset of Y, as required.

□

3. Computing distances in Y

Definition 3.1. A nontrivial freely reduced word $z \in F(a,b,c)$ is said to be a *$\mathcal{W}$-word* if for some $n \geq 1$ and some integer $m \neq 0$ the word z is a subword of w_n^m.

For a freely reduced word $w \in F(a,b,c)$, a *$\mathcal{W}$-decomposition* of w is a decomposition

$$w \equiv z_1 \dots z_k$$

such that each z_i is a $\mathcal{W}$-word.

Remark 3.2. Note that since each of the positive words $v_n(a,b)$ is 7-aperiodic and $|v_n| \to \infty$ as $n \to \infty$, each of the letters a, b appears in v_n for all sufficiently large n. Also, by definition $w_n = v_n c$. Hence every letter from $\{a,b,c\}^{\pm 1}$ is a $\mathcal{W}$-word.

Let Z be the set of all positive $\mathcal{W}$-words $z \in F(a,b,c)$. Then the graph Y can also be viewed as the Cayley graph of $F(a,b,c)$ with respect to the generating set Z.

Lemma 3.3. *Let $z(a,b) \in F(a,b)$ be a nontrivial freely reduced word. Then z is a $\mathcal{W}$-word if and only if there is $n \geq 1$ such that z is a subword of v_n or of v_n^{-1}.*

Proof. If $z(a,b)$ is a $\mathcal{W}$-word and thus a subword of some $w_n^m = (v_n(a,b)c)^m$ (where $m \in \mathbb{Z} \setminus \{0\}$) then, since z does not involve $c^{\pm 1}$ it follows that z is a subword of v_n or of v_n^{-1}. The statement of the lemma now follows.

□

Notation 3.4. For $g \in F(a,b,c)$ denote $|g|_Y := d_Y(1,g)$.

Lemma 3.5 (Distance formula)**.** *Let $w \in F(a,b,c)$ be a nontrivial freely reduced word.*

Then $|w|_Y$ is equal to the smallest $k \geq 1$ such that there exists a $\mathcal{W}$-decomposition $w \equiv z_1 \dots z_k$.

Proof. The definition of Y implies that if $z \in F(a,b,c)$ is a $\mathcal{W}$-word, then for every $g \in F(a,b,c)$ we have $d_Y(g, gz) = 1$. Thus if $w \equiv z_1 \dots z_t$ is a $\mathcal{W}$-decomposition, then $|w|_Y \le t$.

Suppose now that $\gamma = e_1 e_2 \dots e_k$ is a d_Y-geodesic edge-path from 1 to w in Y, where $k = |w|_Y$. Put $u_i = \mu(e_i) \in F(a,b,c)$. Then $w =_{F(a,b,c)} u_1 u_2 \dots u_k$, and each u_i is a $\mathcal{W}$-word.

After freely reducing the product $u_1 u_2 \dots u_k$ we get a factorization $w \equiv z_1 \dots z_r$ where $r \le k$ and each z_i is the remainder of exactly one of the u_j after all the free cancelations are performed. Thus each z_i is a $\mathcal{W}$-word as well, and $w \equiv z_1 \dots z_r$ is a $\mathcal{W}$-decomposition. Hence, by the argument above, $k = |w|_Y \le r$. Thus $k = r$ and we have found a $\mathcal{W}$-decomposition $w \equiv z_1 \dots z_k$ with $k = |w|_Y$.

We have already seen that if w has a $\mathcal{W}$-decomposition with t factors, then $|w|_Y \le t$.

Therefore $|w|_Y$ is equal to the smallest number of factors among all $\mathcal{W}$-decompositions of w, as required. □

Proposition 3.6. *Let $1 \ne g \in F(a,b)$ be arbitrary. Then:*

1. *For every $n \ge 1$ we have $|g^n|_Y \ge \lfloor \frac{n}{7} \rfloor$.*
2. *We have $||g||_Y \ge \frac{1}{7}$.*

Proof. Let $g \equiv uwu^{-1}$ where $u, w \in F(a,b)$ are freely reduced and w is cyclically reduced. Then the freely reduced form of g^n is $uw^n u^{-1}$.

Let $uw^n u^{-1} \equiv z_1 \dots z_k$ be a $\mathcal{W}$-factorization of the word $uw^n u^{-1}$. Thus each z_i is a $\mathcal{W}$-word and $z_i \in F(a,b)$. Hence by Lemma 3.3, each z_i is a subord of some $v_{n_i}^{\pm 1}$. Since the words $v_j(a,b)$ are 7-aperiodic, it follows that for every subword of $uw^n u^{-1}$ of the form w^7 this subword nontrivially overlaps at least two distinct factors z_i. Therefore $k \ge \lfloor \frac{n}{7} \rfloor$.

Hence, by the distance formula provided by Lemma 3.5, for every $n \ge 1$ we have $|g^n|_Y \ge \lfloor \frac{n}{7} \rfloor$. The definition of $||g||_Y$ now implies that $||g||_Y \ge \frac{1}{7}$. □

4. Acylindricity

The following useful fact is a special case of Lemma 4 of Lyndon-Schützenberger [27]:

Lemma 4.1. *Let $u_1, u_2 \in F(a,b,c)$ be nontrivial cyclically reduced words such that for some $k, t \ge 1$ the words u_1^k and u_2^t have a common initial segment of length $\ge |u_1| + |u_2|$. Then there exists a unique root-free cyclically reduced word $u_0 \in F(a,b,c)$ such that $u_1 \equiv u_0^r$ and $u_2 \equiv u_0^s$ for some $r, s \ge 1$.*

Lemma 4.2. *Let $R \ge 1$ and let $L \ge 100(R + 4C)(R + 6C + 10)$.*

Let $h \in F(a,b,c)$ be a freely reduced word and let $g \equiv \alpha^{-1} u \alpha \in F(a,b,c)$ be a freely reduced word with u being cyclically reduced.

Suppose that $|h|_Y \ge L$, $|g|_Y \le R$ and $|hgh^{-1}|_Y \le R$. Then $h \equiv h_0 \sigma_1 \sigma_2 u^k \alpha$ where:

1. *We have $|k| \ge 100(R + 6C + 1)$.*
2. *σ_1, σ_2 are subwords of $\alpha^{-1} u^{\pm 1} \alpha$.*
3. *We have $|h_0|_Y, |\sigma_1|_Y, |\sigma_2|_Y \le R + 4C$.*

Proof. Let $k \in \mathbb{Z}$ be the largest in the absolute value integer such that the freely reduced word $h \in F(a,b,c)$ ends in $u^k \alpha$, where $k = 0$ corresponds to the case where h does not end in $u^{\pm 1} \alpha$. It is not hard to see, by a variation of the argument below, that $k = 0$ is not possible under the assumptions of this lemma, so we can write h as $h \equiv h_1 u^k \alpha$. We will assume that $k > 0$ as the case $k < 0$ is similar.

Then, at the level of group elements, in $F(a,b,c)$ we have

$$hgh^{-1} = h_1 \alpha (\alpha^{-1} u^k \alpha)(\alpha^{-1} u \alpha)(\alpha^{-1} u^{-k} \alpha) \alpha^{-1} h_1^{-1} = h_1 \alpha (\alpha^{-1} u \alpha) \alpha^{-1} h_1^{-1}.$$

Put $h_2 = h_1 \alpha \in F(a,b,c)$, so that h_2 is a freely reduced word. The maximal choice of k implies that in freely reducing the product $h_2 \cdot (\alpha^{-1} u \alpha) \cdot h_2^{-1}$ not all of the word $\alpha^{-1} u \alpha$ cancels. Hence the freely reduced form of hgh^{-1} is graphically equal to $h_3 u_1 h_4^{-1}$ where u_1 is a subword of $\alpha^{-1} u \alpha$, where $h_2 \equiv h_3 \tau$ with τ^{-1} being an initial segment of $\alpha^{-1} u \alpha$ and where $h_2 \equiv h_4 \nu$ with ν^{-1} being a terminal segment of $\alpha^{-1} u \alpha$. We can express $h_1 \equiv h_5 \rho$, where ρ^{-1} is a maximal initial segment of α that cancels in the product $h_1 \alpha$,

with $\alpha \equiv \rho^{-1}\alpha_1$. Then $h_2 \equiv h_5\alpha_1 \equiv h_3\tau$ and $h_2 \equiv h_5\alpha_1 \equiv h_4\nu$. Recall also that the freely reduced form of hgh^{-1} is graphically equal to $h_3u_1h_4^{-1}$. Hence there exist subwords $\sigma_1, \dots, \sigma_4$ and $\beta_1, \dots, \beta_4$ of $\alpha^{-1}u^{\pm 1}\alpha$ such that $h_1 \equiv h_6\sigma_1\sigma_2 \equiv h_7\sigma_3\sigma_4$ such that the freely reduced form of hgh^{-1} is graphically equal to $h_6\beta_1\beta_2u_1\beta_3^{-1}\beta_4^{-1}h_7^{-1}$. Recall also that u_1 is a subword of $\alpha^{-1}u\alpha$ and that $h \equiv h_1u^k\alpha$.

By assumption, $|hgh^{-1}|_Y \leq R$. Since the freely reduced form of hgh^{-1} is $h_6\beta_1\beta_2u_1\beta_3^{-1}\beta_4^{-1}h_7^{-1}$, Corollary 2.6 implies that $|h_6|_Y, |h_7|_Y \leq R + 4C$. Since $\sigma_1, \sigma_2, \alpha$ are subwords of the freely reduced word $g = \alpha^{-1}u\alpha$, and since by assumption $|g|_Y \leq R$, Corollary 2.6 implies that $|\sigma_1|_Y, |\sigma_2|_Y, |\alpha|_Y \leq R + 4C$. We also have $h \equiv h_1u^k\alpha \equiv h_6\sigma_1\sigma_2u^k\alpha$, and by assumption $|h|_Y \geq L$. By the triangle inequality we now get $|u^k|_Y \geq L - 4(R + 4C)$. Since $|g|_Y \leq R$, Corollary 2.6 implies that $|u|_Y \leq R + 4C$. Thus

$$L - 4(R + 4C) \leq |u^k|_Y \leq k(R + 4C)$$

and hence $k \geq (L - 4(R + 4C))/(R + 4C) = \frac{L}{R+4C} - 4 \geq 100(R + 6C + 1)$, where the last inequality holds by the assumption on L. Thus the factorization $h \equiv h_6\sigma_1\sigma_2u^k\alpha$ satisfies all the requirements of the lemma. $\square$

Proposition 4.3. *Let $R \geq 1$ and $L \geq 100(R + 4C)(R + 6C + 10)$. Let $g, g' \in F(a, b, c)$ be nontrivial freely reduced words conjugate in $F(a, b, c)$ to some elements of $F(a, b)$, and let $h \in F(a, b, c)$ be such that $|h|_Y \geq L$, $|g|_Y, |g'|_Y \leq R$ and that $d_Y(h, gh), d_Y(h, g'h) \leq R$. Then there exists a root-free nontrivial freely reduced $g_0 \in F(a, b, c)$ such that $g = g_0^t$, $g' = g_0^r$, where $1 \leq |r|, |t| \leq 7(R + 4C + 1)$.*

Proof. We have $d_Y(h, gh) = |h^{-1}gh|_Y, d_Y(h, g'h) = |h^{-1}g'h| \leq R$. Write g as a freely reduced word $g \equiv \alpha^{-1}u\alpha \in F(a, b)$, with $u \in F(a, b)$ being cyclically reduced. Similarly, write g' as a freely reduced word $g' \equiv (\alpha')^{-1}u'\alpha' \in F(a, b)$, with $u' \in F(a, b)$ being cyclically reduced.

Applying Lemma 4.2 we conclude that there exist factorizations $h^{-1} \equiv h_0\sigma_1\sigma_2u^k\alpha$ and $h^{-1} \equiv h_0'\sigma_1'\sigma_2'(u')^r\alpha'$ where $|k|, |r| \geq 100(R + 6C + 1)$, where σ_1, σ_2 are subwords of g, where where σ_1', σ_2' are subwords of g', and where $|h_0|_Y, |h_0'|_Y, |\sigma_1|_Y, |\sigma_2|_Y, |\sigma_1'|_Y, |\sigma_2'| \leq R + 4C$.

We now see how the subwords u^k and $(u')^s$ overlap in

$$h^{-1} \equiv h_0\sigma_1\sigma_2u^k\alpha \equiv h_0'\sigma_1'\sigma_2'(u')^s\alpha'.$$

Case 1. Suppose first that the length of the overlap between u^k and $(u')^s$ is $< |u| + |u'|$. Without loss of generality we may assume that $|u'| \leq |u|$ and that $k, r > 0$.

Then either u^{k-2} is a subword of $h_0'\sigma_1'\sigma_2'$, or u^{k-2} is a subword of α', or $(u')^r$ is contained in u^k.

Recall that $k, r \geq 100(R + 6C + 1)$.

If u^{k-2} is a subword of $h_0'\sigma_1'\sigma_2'$ then Corollary 2.6 implies that $|u^{k-2}|_Y \leq |h_0'\sigma_1'\sigma_2'|_Y + 4C \leq 3(R + 4C) + 4C = 3R + 16C$. Since $u \in F(a, b)$, Proposition 3.6 implies that $|u^{k-2}|_Y \geq (k - 2)/7 - 1$. Hence $(k - 2)/7 - 1 \leq |u^{k-2}|_Y \leq 3R + 16C$ and $k \leq 7(3R + 16C + 1) + 2$, yielding a contradiction.

If u^{k-2} is a subword of α', then Corollary 2.6 implies that $|u^{k-2}|_Y \leq |\alpha'|_Y + 4C \leq R + 6C$. Since $|u^{k-2}|_Y \geq (k - 2)/7 - 1$, we get $(k - 2)/7 - 1 \leq |u^{k-2}|_Y \leq R + 6C$ and $k \leq 7(R + 6C + 1) + 2$, again yielding a contradiction with $k \geq 100(R + 6C + 1)$.

Suppose now that $(u')^r$ is contained in u^k. Since $|u'| \leq |u|$ and the length of the overlap between u^k and $(u')^r$ is $< |u| + |u'|$, it follows that $(u')^r$ is contained in some subword u^2 or u^k. Hence either $u^{k/4}$ is a subword of $h_0'\sigma_1'\sigma_2'$ or $u^{k/4}$ is a subword of α'. We then again obtain a contradiction by a similar argument to above.

Case 2. Suppose now that the length of the overlap between u^k and $(u')^s$ is $\geq |u| + |u'|$. Without loss of generality we may assume that $|\alpha| \leq |\alpha'|$.

Assume first that $|\alpha| = |\alpha'|$, so that $\alpha' = \alpha$. Then Lemma 4.1 implies that there exists a cyclically reduced word $u_0 \in F(a, b)$ such that $u = u_0^t$ and $u' = u_0^s$, so that $g = (\alpha^{-1}u_0\alpha)^t$ and $g' = (\alpha^{-1}u_0\alpha)^s$. By assumption $|g|_Y, |g'|_Y \leq R$ which by Corollary 2.6 implies that $|u_0^t|_Y, |u_0^s|_Y \leq R + 4C$. Hence by Proposition 3.6 we have $|t|/7 - 1, |s|/7 - 1 \leq R + 4C$ and hence $|t|, |s| \leq 7(R + 4C + 1)$, as required. The conclusion of the proposition is established in this case.

Assume now that $|\alpha| < |\alpha'|$. Let u_* be the cyclic permutation of u such that the overlap between u^k and $(u')^r$ in h_1^{-1} ends in u_*. Lemma 4.1 implies that there exists a cyclically reduced word $u_0 \in F(a, b)$ such

that $u_* = u_0^t$ and $u' = u_0^s$. We may assume (after possibly replacing u_0 by its inverse) that $rs > 0$. The fact that $|\alpha| < |\alpha'|$ now implies that the first letter of α' is the same as the first letter of u_0. This contradicts the fact that the word $g' \equiv (\alpha')^{-1}(u')^r\alpha = (\alpha')^{-1}u_0^{rs}\alpha'$ is freely reduced as written. Thus Case 2 cannot happen, which completes the proof of the proposition.

□

Corollary 4.4. *The action of $F(a,b)$ on Y is acylindrical.*

Proof. It is enough to check the acylindricity condition for the vertices of Y.

Let $R \geq 1$. Put $L = L(R) := 100(R+4C)(R+6C+10)$ and $M = M(R) := 14(R+4C+1)+1$.

Let $x, y \in VY = F(a,b,c)$ be vertices such that $d_Y(x,y) \geq L$. Put

$$S = \{g \in F(a,b) | d_Y(x,gx) \leq R, d_Y(y,gy) \leq R\}.$$

We claim that $\#(S) \leq M$.

We have $d_Y(x,y) = d_Y(1, x^{-1}y) \geq L$. Let $g \in F(a,b)$ be such that $d_Y(x,gx) \leq R, d_Y(y,gy) \leq R$. Then for $g_1 = x^{-1}gx$ we have $|g_1|_Y = |x^{-1}gx|_Y = d_Y(x,gx) \leq R$ and

$$d_Y(x^{-1}y, g_1x^{-1}y) = |y^{-1}x^{-1}g_1x^{-1}y|_Y = |y^{-1}x^{-1}x^{-1}gxx^{-1}y|_Y = |y^{-1}gy|_Y = d_Y(y,gy) \leq R.$$

Put $h = x^{-1}y \in F(a,b,c)$, so that $|h|_Y = d_Y(x,y) \geq L$. Also put

$$S_1 := \{g_1 \in F(a,b,c) \big| \\ |g_1|_Y \leq R, |h^{-1}g_1h|_Y \leq R, \text{ and } g_1 \text{ is conjugate to an element of } F(a,b) \text{ in } F(a,b,c)\}.$$

Since $x^{-1}Sx \subseteq S_1$, to verify the claim above it is enough to show that $\#(S_1) \leq M$.

Suppose $\#(S_1) \geq 2$. Let $1 \neq g_1 \in S_1$. We can uniquely express g_1 as $g_1 = g_0^t$ where $g_0 \in F(a,b,c)$ is a nontrivial root-free element and $t \geq 1$. Now if $g_2 \in S_1$ is an arbitrary nontrivial element, then Proposition 4.3 implies that $g_2 = g_0^s$ where $|s| \leq 7(R+4C+1)$. It follows that $\#(S_1) \leq M$, as required.

□

We now summarize the properties of the action of $F(a,b)$ on Y:

Theorem 4.5. *The following hold:*

1. *The graph Y is Gromov-hyperbolic and $F(a,b)$ acts on Y by simplicial isometries.*
2. *The action of $F(a,b)$ on Y is acylindrical.*
3. *The action of $F(a,b)$ on Y is purely loxodromic, that is, every $1 \neq g \in F(a,b)$ acts on Y as a loxodromic isometry.*
4. *For every $1 \neq g \in F(a,b)$ we have $||g||_Y \geq 1/7$.*
5. *For any $p \in Y$ the orbit $F(a,b)p \subseteq Y$ is a quasiconvex subset of Y.*
6. *There exists $C \geq 1$ such that for any $x, y \in F(a,b)$ if $\alpha_{x,y}$ is a geodesic from x to y in the Cayley graph of $F(a,b)$ with respect to the basis $\{a,b\}$, and if $\beta = [x,y]_Y$ is a geodesic from x to y in Y, then α and β are C-Hausdorff close in Y.*
7. *For any $p \in Y$, the orbit map $F(a,b) \to Y$, $g \mapsto gp$, is not a quasi-isometric embedding. Moreover, the action of $F(a,b)$ on Y is not proper.*

Proof. Parts (1) and (6) are established in Proposition 2.4. Part (2) is Corollary 4.4 above. Part (4) is Proposition 3.6, and part (4) directly implies part (3). Part (5) is Proposition 2.7.

To see that (7) holds, note that for every $n \geq 1$ $v_n(a,b)$ is a $\mathcal{W}$-word and hence, by definition of Y, we have $d_Y(1, v_n) = |v_n(a,b)|_Y = 1$. On the other hand, v_n is a freely reduced word in $F(a,b)$ with $|v_n| \to \infty$ as $n \to \infty$. This shows, with $p = 1 \in VY$, that the orbit map $F(a,b) \to Y, g \mapsto gp$ is not a quasi-isometric embedding and that the action of $F(a,b)$ on Y is not proper. □

References

[1] E. Alibegović, *A combination theorem for relatively hyperbolic groups.* Bull. London Math. Soc. **37** (2005), no. 3, 459–466

[2] M. Bestvina and M. Feighn, *Hyperbolicity of the complex of free factors.* Adv. Math. **256** (2014), 104–155

[3] M. Bestvina, and M. Handel, *Train tracks and automorphisms of free groups.* Ann. of Math. (2) **135** (1992), no. 1, 1–51

[4] M. Bestvina and M. Feighn, *Outer limits*, preprint, 1994; http://andromeda.rutgers.edu/~feighn/papers/outer.pdf

[5] M. Bestvina, M. Feighn, and M. Handel, *Laminations, trees, and irreducible automorphisms of free groups*, Geometric and Functional Analysis **7** (1997), no. 2, 215–244

[6] B. H. Bowditch, *Tight geodesics in the curve complex.* Invent. Math. **171** (2008), no. 2, 281–300

[7] B. H. Bowditch, *Relatively hyperbolic groups.* Internat. J. Algebra Comput. **22** (2012), no. 3

[8] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999

[9] F. Dahmani, V. Guirardel and D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Memoirs of AMS, to appear; arXiv:1111.7048

[10] T. Delzant, *Sur l'accessibilité acylindrique des groupes de présentation finie.* Ann. Inst. Fourier (Grenoble) **49** (1999), no. 4, 1215–122

[11] S. Dowdall and S. Taylor, *Hyperbolic extensions of free groups*, preprint, 2014; arXiv:1406.2567

[12] S. Dowdall and S. Taylor, *Geometric properties of hyperbolic free group extensions*, in preparation

[13] B. Farb and L. Mosher, *Convex cocompact subgroups of mapping class groups.* Geom. Topol. **6** (2002), 91–152

[14] D. Gaboriau, A. Jaeger, G. Levitt, M. Lustig, *An index for counting fixed points of automorphisms of free groups*, Duke Math. J. **93** (1998), no. 3, 425–452

[15] D. Gruber, and A. Sisto, *Infinitely presented graphical small cancellation groups are acylindrically hyperbolic*, arXiv:1408.4488

[16] U. Hamenstädt, *Word hyperbolic extensions of surface groups*, preprint, 2005; arXiv:math.GT/050524

[17] U. Hamenstadt and S. Hensel, *Convex Cocompact Subgroups of* $\mathrm{Out}(F_n)$, preprint, arXiv:1411.2281

[18] A. Hilion and C. Horbez, *The hyperbolicity of the sphere complex via surgery paths*, Crelle's Journal, to appear; arXiv:1210.6183

[19] M. Hull, *Small cancellation in acylindrically hyperbolic groups*, preprint; arXiv:1308.4345

[20] I. Kapovich, and K. Rafi, *On hyperbolicity of free splitting and free factor complexes.* Groups Geom. Dyn. **8** (2014), no. 2, 391–414

[21] I. Kapovich, and R. Weidmann, *Acylindrical accessibility for groups acting on R-trees*, Math. Z. **249** (2005), no. 4, 773–782

[22] R. P. Kent, and C. J. Leininger, *Subgroups of mapping class groups from the geometrical viewpoint.* In the tradition of Ahlfors-Bers. IV, 119–141, Contemp. Math., 432, Amer. Math. Soc., Providence, RI, 2007

[23] R. P. Kent, and C. J. Leininger, *Shadows of mapping class groups: capturing convex cocompactness*, Geom. Funct. Anal. **18** (2008), 1270–1325.

[24] S. Kim and T. Koberda, *Embedability between right-angled Artin groups*, Geom. Topol. **17** (2013), no. 1, 493–530

[25] T. Koberda, J. Mangahas, and S. J. Taylor, *The geometry of purely loxodromic subgroups of right-angled Artin groups,* preprint; arXiv:1412.3663

[26] G. Levitt, and M. Lustig, *Irreducible automorphisms of* F_n *have north-south dynamics on compactified outer space.* J. Inst. Math. Jussieu **2** (2003), no. 1, 59–72

[27] R. Lyndon and M. Schützenberger, *The equation* $a^M = b^N c^P$ *in a free group*, Michigan Math. J. **9** (1962), 289–298

[28] B. Mann, *Some hyperbolic* $Out(F_N)$*-graphs and nonunique ergodicity of very small* F_N*-trees.* PhD thesis, University of Utah, 2014

[29] H. Masur and Y. Minsky. *Geometry of the complex of curves. I. hyperbolicity.* Invent. Math., **138** (1999), no. 1, 103–149

[30] A. Minasyan, D. Osin, *Acylindrically hyperbolic groups acting on trees*, Math. Ann., to appear; arXiv:1310.6289

[31] A. Ol'sahnskii, *Embedding construction based on amalgamations of group relators*, arXiv:1406.0336; to appear in J. Topol. Anal.

[32] D. Osin, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc., to appear; arXiv:1304.1246

[33] P. Przytycki, and A. Sisto, *A note on acylindrical hyperbolicity of Mapping Class Groups*, preprint; arXiv:1502.02176

[34] Z. Sela, *Acylindrical Accessibility.* Invent. Math. **129** (1997), no. 3, 527–565

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801
http://www.math.uiuc.edu/~kapovich,
E-mail address: kapovich@math.uiuc.edu